

Mandelstam-Leibbrandt prescription

J. Alfaro

Facultad de Física, Pontificia Universidad Católica de Chile,
Casilla 306, Santiago 22, Chile.

jalfaro@uc.cl

March 22, 2016

Abstract

The light cone gauge is used frequently in string theory as well as gauge theories and gravitation. Loop integrals however have to be infrared regulated to remove spurious poles. The most popular and consistent of these infrared regulators is the Mandelstam-Leibbrandt(ML) prescription. The calculations with ML are rather cumbersome, though. In this work we show that the ML can be replaced by a symmetry of the regulator. This symmetry simplify the calculations, reducing them to conventional dimensional regularization integrals.

1 Introduction

Computations in superstring theory as well as gauge theories, supersymmetry, gravitation and Chern-Simons theories are often simplified by recurring to the light cone gauge. The light cone gauge is termed one of the physical gauges because ghosts decouple in these gauges¹. To compute loop corrections in the light cone gauge has some peculiarities, though: Spurious infrared poles appear, non local terms are present and Lorentz invariance is explicitly broken. To deal with these problems an infrared regulator is needed. The most popular and internally consistent regulator used at present is the Mandelstam-Leibbrandt regulator (ML)[2, 3]. ML has very nice properties: The poles in the k_0 complex plane are situated such that the Wick's rotation from Euclidean to Minkowsky space is justified; it preserves naive power counting of loop integrals; and in gauge theories, it maintains the Ward identities of the gauge symmetry[1, 4].

Explicit computations with the ML are long and cumbersome, though.

Here we present a method to evaluate the loop integrals that appear in the light cone gauge based on a scale symmetry of the regulator. No new integrals are required, aside from the standard dimensionally regularized integrals. In fact the ML prescription can be safely replaced by the scale symmetry and a regularity condition. We do not have to specify the exact value of the two null

¹There are some subtleties related to this point. Please see [1] chapter 4.4.

vectors of the ML, but merely its mutual relations. The results coincide with the one obtained with ML.

2 The new prescription

Let us compute the following simple integral:

$$A_\mu = \int dp \frac{f(p^2)p_\mu}{(n \cdot p)}$$

where f is an arbitrary function. dp is the integration measure in d dimensional space and n_μ is a fixed null vector ($n \cdot n = 0$). This integral is infrared divergent when $(n \cdot p) = 0$.

The ML is:

$$\frac{1}{(n \cdot p)} = \lim_{\varepsilon \rightarrow 0} \frac{(p \cdot \bar{n})}{(n \cdot p)(p \cdot \bar{n}) + i\varepsilon} \quad (1)$$

where \bar{n}_μ is a new null vector with the property $(n \cdot \bar{n}) = 1$.

To compute A_μ we have to know the specific form of f , provide an specific form of n_μ and \bar{n}_μ , and evaluate the residues of all poles of $\frac{f(p^2)}{(n \cdot p)}$ in the p_0 complex plane, a rather formidable task for an arbitrary f .

Instead we want to point out the following symmetry:

$$n_\mu \rightarrow \lambda n_\mu, \bar{n}_\mu \rightarrow \lambda^{-1} \bar{n}_\mu, \lambda \neq 0, \lambda \varepsilon R \quad (2)$$

It preserves the definitions of n_μ and \bar{n}_μ :

$$\begin{aligned} 0 &= (n \cdot n) \rightarrow \lambda^2 (n \cdot n) = 0 \\ 0 &= (\bar{n} \cdot \bar{n}) \rightarrow \lambda^{-2} (\bar{n} \cdot \bar{n}) = 0 \\ 1 &= (n \cdot \bar{n}) \rightarrow (n \cdot \bar{n}) = 1 \end{aligned}$$

We see from (1) that:

$$\frac{1}{(n \cdot p)} \rightarrow \frac{1}{(n \cdot p)} \lambda^{-1}$$

Now we compute A_μ , based on its symmetries. It is a Lorentz vector which scales under (2) as λ^{-1} . The only Lorentz vectors we have available in this case are n_μ and \bar{n}_μ . But (2) forbids n_μ . That is:

$$A_\mu = a \bar{n}_\mu$$

Multiply by n_μ to find $(A \cdot n) = a$. Thus $a = \int dp f(p^2)$. Finally:

$$\int dp \frac{f(p^2)p_\mu}{(n \cdot p)} = \bar{n}_\mu \int dp f(p^2)$$

By the same token we find

$$A_{\mu\nu\lambda} = \int dp \frac{f(p^2)p_\mu p_\nu p_\lambda}{(n \cdot p)} = a(\bar{n}_\mu g_{\nu\lambda})_S + b(\bar{n}_\mu \bar{n}_\nu n_\lambda)_S$$

where $()_S$ means symmetric in all Lorentz indices.

We get:

$$A_{\mu\nu\lambda}n^\lambda = \frac{1}{d}g_{\mu\nu} \int dp f(p^2)p^2$$

Therefore

$$a = \frac{1}{d} \int dp f(p^2)p^2 = -b$$

The integrals on p_μ are dimensionally regularized.

Therefore:

$$\int dp \frac{f(p^2)p_\mu p_\nu p_\lambda}{(n \cdot p)} = \frac{1}{d} \int dp f(p^2)p^2 \{(\bar{n}_\mu g_{\nu\lambda})_S - (\bar{n}_\mu \bar{n}_\nu n_\lambda)_S\}$$

2.1 Generic integrals

We consider now a more general integral. We will see here that regularity of the answer will determine it uniquely.

Consider:

$$A = \int dp \frac{F(p^2, p \cdot q)}{(n \cdot p)} = (\bar{n} \cdot q) f(q^2, (n \cdot q)(\bar{n} \cdot q)) \quad (3)$$

q_μ is an external momentum, a Lorentz vector. F is an arbitrary function. The last relation follows from (2), for a certain f we will find in the following.

We get

$$\begin{aligned} \frac{\partial A}{\partial q^\mu} &= \int dp \frac{F_{,u} p_\mu}{(n \cdot p)} = \\ &= \bar{n}_\mu f(x, y) + 2(\bar{n} \cdot q) q_\mu \frac{\partial}{\partial x} f(x, y) + [(\bar{n} \cdot q)^2 n_\mu + (n \cdot q)(\bar{n} \cdot q) \bar{n}_\mu] \frac{\partial}{\partial y} f(x, y) \end{aligned}$$

We defined $u = p \cdot q, x = q^2, y = (n \cdot q)(\bar{n} \cdot q)$. $()_{,u}$ means derivative respects to u .

$$\begin{aligned} \frac{\partial A}{\partial q^\mu} n_\mu &= \int dp F_{,u} = g(x) = \\ &= f(x, y) + 2y \frac{\partial}{\partial x} f(x, y) + y \frac{\partial}{\partial y} f(x, y) \end{aligned} \quad (4)$$

Assuming that the solution and its partial derivatives are finite in the neighborhood of $y = 0$, it follows from the equation that $f(x, 0) = g(x)$. That is the partial differential equation has a unique regular solution.

We will find the solution of (4) using the method of characteristics[5].

$$\begin{aligned} \dot{x} &= 2y & \dot{y} &= y \\ \dot{f} + f &= g(x(t)) \\ y &= C e^t & \dot{x} &= 2C e^t, x = 2C e^t + D \\ x - 2y &= D \end{aligned}$$

The most general solution of the system is:

$$f = be^{-t} + e^{-t} \int_{-\infty}^t dt' e^{t'} g(x(t'))$$

for b arbitrary corresponding to the solution of (4) with $g = 0$ (homogeneous solution). The regular solution of (4), f_0 , is obtained imposing that $b = 0$. The reason being that the homogeneous solution is: $f = \Pi(x - 2y)y^{-1}$, with Π an arbitrary function. We readily see that f will diverge at $y = 0$, unless $\Pi(x) = 0$, for all x .

Moreover

$$\lim_{t \rightarrow -\infty} f_0 = \lim_{t \rightarrow -\infty} e^{-t} \int_{-\infty}^t dt' e^{t'} g(x(t')) = g(D)$$

That is $f_0(x, 0) = g(x)$. f_0 is the unique regular solution of (4).

What we have developed up to here shows that the scale transformation (2) plus the regularity condition determines uniquely the value of the integral (3).

2.2 Application to loop integrals

We consider now integrals that appear in gauge theory loops:

$$\int dp \frac{1}{[p^2 + 2p \cdot q - m^2]^a} \frac{1}{(n \cdot p)} = (\bar{n} \cdot q) f(x, y)$$

In this case

$$g(x) = -2a \int dp \frac{1}{[p^2 - x - m^2]^{a+1}}$$

Therefore

$$\begin{aligned} f &= e^{-t} \int_{-\infty}^t dt' e^{t'} g(x(t')) = \\ &= \int dp e^{-t} [p^2 - 2Ce^{t'} - D - m^2]^{-a} \frac{1}{-C} \Big|_{-\infty}^t = \\ &= -\frac{1}{y} \left\{ \int dp [p^2 - x - m^2]^{-a} - \int dp [p^2 - x + 2y - m^2]^{-a} \right\} \end{aligned}$$

We readily verify that $f(x, 0) = -2a \int dp [p^2 - x - m^2]^{-a-1} = g(x)$

In the same way we get:

$$\int dp \frac{1}{[p^2 + 2p \cdot q - m^2]^a} \frac{1}{((n \cdot p))^2} = (\bar{n} \cdot q)^2 f(x, y)$$

with

$$f(x, y) = \frac{1}{y^2} \int dp \{ [p^2 - x - m^2]^{-a} - [p^2 - x + 2y - m^2]^{-a} - 2ay [p^2 - x + 2y - m^2]^{-a-1} \}$$

Following the same procedure we can get an answer for the whole family of loop integrals:

$$\int dp \frac{1}{[p^2 + 2p \cdot q - m^2]^a} \frac{1}{((n \cdot p))^b} = (\bar{n} \cdot q)^b (-2)^b \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dt t^{b-1} \int dp [p^2 - q^2 + 2n \cdot q \bar{n} \cdot q t - m^2]^{-a-b}$$

Using dimensional regularization, we obtain:

$$\int dp \frac{1}{[p^2 + 2p \cdot q - m^2]^a} \frac{1}{((n \cdot p))^b} = (-1)^{a+b} i(\pi)^\omega (-2)^b \frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)} (\bar{n} \cdot q)^b \int_0^1 dt t^{b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega}}, \omega = d/2 \quad (5)$$

We sketch the proof of equation (5).

$$\int dp \frac{1}{[p^2 + 2p \cdot q - m^2]^a} \frac{1}{((n \cdot p))^b} = (\bar{n} \cdot q)^b f(b, a, x, y)$$

with

$$-2af(b-1, a+1, x, y) = bf(b, a, x, y) + 2y \frac{\partial}{\partial x} f(b, a, x, y) + y \frac{\partial}{\partial y} f(b, a, x, y), \quad (6)$$

$$f(b, a, x, 0) = -\frac{2a}{b} f(b-1, a+1, x, 0) \quad (7)$$

It is easy to check that (5) satisfies the partial differential equation (6) and the boundary condition (7), so it is the unique regular solution and thus determine the value of the integral.

Other integrals can be obtained deriving respects to q_μ :

$$\begin{aligned} & \int dp \frac{p_\mu}{[p^2 + 2p \cdot q - m^2]^{a+1}} \frac{1}{((n \cdot p))^b} = \\ & (-1)^{a+b} i(\pi)^\omega (-2)^{b-1} \frac{\Gamma(a+b-\omega)}{\Gamma(a+1)\Gamma(b)} (\bar{n} \cdot q)^{b-1} b \bar{n}_\mu \int_0^1 dt t^{b-1} \frac{1}{(m^2 + x - 2yt)^{a+b-\omega}} + \\ & (-1)^{a+b} i(\pi)^\omega (-2)^b \frac{\Gamma(a+b+1-\omega)}{\Gamma(a+1)\Gamma(b)} (\bar{n} \cdot q)^b \int_0^1 dt t^{b-1} \frac{q_\mu - t(n \cdot q \bar{n}_\mu + \bar{n} \cdot q n_\mu)}{(m^2 + x - 2yt)^{a+b+1-\omega}} \end{aligned} \quad (8)$$

and

$$\begin{aligned} & \int dp \frac{p_\mu p_\nu}{[p^2 + 2p \cdot q - m^2]^{a+2}} \frac{1}{((n \cdot p))^b} = (-1)^{a+b} i(\pi)^\omega (-2)^{b-2} \{ \\ & \frac{\Gamma(a+b-\omega)}{\Gamma(a+2)\Gamma(b-1)} (\bar{n} \cdot q)^{b-2} b \bar{n}_\mu \bar{n}_\nu \int_0^1 dt t^{b-1} \frac{1}{(m^2 + x - 2yt)^{a+b-\omega}} \\ & - 2 \frac{\Gamma(a+b+1-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n} \cdot q)^{b-1} b \bar{n}_\mu \int_0^1 dt t^{b-1} \frac{(q_\nu - t(n \cdot q \bar{n}_\nu + \bar{n} \cdot q n_\nu))}{(m^2 + x - 2yt)^{a+b+1-\omega}} \\ & - 2 \frac{\Gamma(a+b+1-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n} \cdot q)^{b-1} b \bar{n}_\nu \int_0^1 dt t^{b-1} \frac{q_\mu - t(n \cdot q \bar{n}_\mu + \bar{n} \cdot q n_\mu)}{(m^2 + x - 2yt)^{a+b+1-\omega}} \\ & + 4 \frac{\Gamma(a+b+2-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n} \cdot q)^b \int_0^1 dt t^{b-1} \frac{[q_\mu - t(n \cdot q \bar{n}_\mu + \bar{n} \cdot q n_\mu)][q_\nu - t(n \cdot q \bar{n}_\nu + \bar{n} \cdot q n_\nu)]}{(m^2 + x - 2yt)^{a+b+2-\omega}} \} \quad (9) \end{aligned}$$

The right hand side of equations (5,8,9) is analytic in the parameters a, b, ω almost everywhere in their respective complex planes, so it provides the analytic extension of the integral to these wider domains.

3 Comparison with ML

The simpler integral $A_\mu, A_{\mu\nu\lambda}$ of section 2, agree with the ML prescriptions[1].

But, in this section we want to compute a more involved integral, in order to compare both finite and divergent results with ML's.

We want to compute:

$$A(\sigma, q) = \int dp \frac{(p^2)^{\sigma-1}}{(p-q)^2((n \cdot p))^2}$$

We introduce Feynman parameters [6]

$$\frac{1}{A_1^{m_1} A_2^{m_2}} = \int_0^1 dx \frac{x^{m_1-1} (1-x)^{m_2-1}}{[xA_1 + (1-x)A_2]^{m_1+m_2}} \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_2)}$$

to get,

$$A(\sigma, q) = (1-\sigma) \int_0^1 dx \int dp \frac{x^{-\sigma}}{[p^2 + (1-x)(-2p \cdot q + \bar{q}^2)]^{2-\sigma} ((n \cdot p))^2}$$

Using (5), we finally get

$$A(\sigma, q) = 4(\bar{n} \cdot q)^2 (-1)^{\sigma} i(\pi)^{\omega} \frac{\Gamma(4-\sigma-\omega)}{\Gamma(1-\sigma)} \int_0^1 dx x^{-\sigma} (1-x)^{\sigma+\omega-2} \int_0^1 dt t [q^2 x + 2t(n \cdot q)(\bar{n} \cdot q)(1-x)]^{\sigma+\omega-4}$$

This coincides with [7] equation C4, by the change of variable $t = 1 - y$ and $\sigma = \omega$.

Notice that we have considered $\bar{q}^2 \neq q^2$ and taken the limit $\bar{q}^2 = q^2$ after evaluating the integral. This is justified because the integral is a regular function of q^2 . If we expand $A(\sigma, q)$ in powers of q^2 , each term of the series can be evaluated using (5). The summation of the series is equivalent to the procedure we followed above.

4 Conclusions

We have developed a way of evaluation of the light cone loop integrals based on the scale symmetry (2) and the condition of regularity of the solution. We do not have to specify the exact value of the two null vectors of the ML, but merely its mutual relations. The answer is the same than in the ML prescription, but a significant simplification of the calculation is available now.

For future work, we want to mention that the scale transformation (2) is also a symmetry of the uniform prescription introduced by Leibbrandt[1] to treat the spurious infrared poles in light-cone, axial, planar and temporal gauge. The application of the method presented here to these more general gauges will be done elsewhere.

5 Acknowledgements

The work of J. A. is partially supported by Fondecyt 1150390, CONICYT-PIA-ACT1102 and CONICYT-PIA-ACT1417.

References

- [1] G. Leibbrandt, Quantization of Yang-Mills and Chern-Simons Theory in Axial-type gauges, World Scientific, Singapore 1994.
- [2] S. Mandelstam, Nucl. Phys. B213, 149 (1983).
- [3] G. Leibbrandt, Phys. Rev. D29, 1699 (1984).
- [4] A. Bassetto, G. Nardelli and R. Soldati, Yang-Mills Theories in Algebraic Non-Covariant Gauges (World Scientific, Singapore, 1991).
- [5] E.T. Copson, Partial Differential Equations, Cambridge University Press, 1975.
- [6] M.E. Peskin, D.V. Schroeder, An Introduction to Quantum Field Theory, Perseus, 1995.
- [7] Leibbrandt, G., and S.-L. Nyeo, J. Math. Phys. 27, 627.